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TAGMEMIC ANALYSIS OF ELEMENTARY ALGEBRA

1. INTRODUCTION

Considerable work has been done in applying mathematical techniques to the analysis of language, especially the more formal languages associated with various types of automata.[1] To my knowledge, however, very little has been done in the opposite direction: applying linguistic techniques to the analysis of mathematics as a specialized language occurring in certain textbooks, journal articles, and classroom situations.[2] Yet, since mathematical language is simpler than natural language, one might well expect the direction linguistics-applied-to-mathematics to provide results at least as fruitful as and with less labor than the direction mathematics-applied-to-linguistics.

This paper attempts a preliminary linguistic analysis of material from elementary algebra, with particular attention (§§3-6) to the types of structure above the clause level, and to the semiotic interlocking of mathematical language with ordinary language (see especially §7).

2. THE CLAUSE LEVEL AND BELOW

Consider the solution to a quadratic equation as it might appear in a book on elementary algebra:

1. Problem: solve $x^2 + x - 12 = 0$.
   Solution: $x^2 + x - 12 = 0$
   $(x - 3) \cdot (x + 4) = 0$
   $x - 3 = 0$ or $x + 4 = 0$
   $x = 3$ or $x = -4$.

The mathematical formulas '$x - 3$', '$x = -4$', '$x - 3 = 0$ ', and the like can be represented in tagmemic theory as follows:

2. Formula$_1$ = +ARGUMENT$_1$ element +P-LINK^2 equational
| Term | <=>

+ARGUMENT$_2$ element
| Term

In (2) grammatical slots are represented by upper-case words (‘ARGUMENT$_1$’, ‘P-LINK^2’); sememic slots are represented by lower case words on the same (upper) line of print (‘element’, ‘equational’); the filler classes for these slots are represented by words on the lower line of print (‘Term’); and the syntagmeme combining the slots into a unit on a higher level (‘Formula$_1$’) is represented by the ‘+’s and ‘=’ on the upper line of print.

Thus (2) is to be interpreted as follows: a mathematical formula of the first type (Formula$_1$) is a syntagmeme composed of three immediate constituents. The first constituent must be a term (like ‘x’, ‘x – 3’, or ‘0’) filling an argument slot (the first argument slot, argument *sub-1*); the second constituent must be a member of the class <=> , i.e. must be ‘=’, filling a “p-link^2” slot; and the third constituent, like the first, must be a term filling the second argument slot. Each constituent in the syntagmeme has a sememic meaning indicated by the lower-case letters on the upper line. The first term denotes an “element”, an item in the “real” mathematical model. (In the case of a quadratic equation one may think of the model as the set of complex numbers together with certain standard operations (+, –, ·, /) and relations (=) on it.) The second constituent ‘=’ denotes the identity relation in the model. The third constituent denotes an “element” in the model. All three constituents appear *obligatorily* (indicated by the ‘+’ preceding the slots; optional slots will be indicated by ‘±’), and in the order indicated (a term, followed by ‘=’, followed by a term).

I have coined the term ‘p-link’, or ‘predicate link’, to indicate any grammatical slot filled by ‘=’ or by a predicate-like symbol. The predicate link takes term-like entries for arguments and forms out of them a sentence-like syntagmeme. The superscript ‘^{2}’ of ‘p-link^2’ indicates that this link takes 2 arguments. Likewise we may have not only “p-link^n” with n arguments, but l-links (logical links, cf. (3) below), taking sentence-like entries for arguments and forming from them sentence-like syntagmemes; or f-links (function links, cf. (6) below), taking term-like entries for arguments and forming from them term-like syntagmemes. We thus obtain Figure 1, in which the type of argument taken by the link is indicated in rows and the type of the resulting construction in columns. In theory we might also have g-links

(Gödel links) with sentence-like units as arguments and term-like units as result – Gödel's arithmetization of formulas is a standard example. However, we do not encounter such links in elementary algebra except in the more tacit form of quotation marks, nor do we encounter the even more exotic links which may have sentence-like units for some arguments and term-like units for others.[3] The absence of 'exotic' forms corresponds closely with the preference in natural languages for phrase-out-of-phrases, sentence-out-of-phrases syntagmemes as opposed to phrase-out-of-sentences syntagmemes or phrase-out-of-sentence-plus-phrase syntagmemes.

FIGURE 1

Links

		resulting constructions →	
		term-like	sentence-like
	term-like	F-LINK (function link)	P-LINK (predicate link)
arguments ↓	sentence-like	(G-LINK?)	L-LINK (logical link)
	term-like and/ or sentence-like	–	–

To return to the quadratic equation (1). How shall we deal with the logical construction with 'or' in 'x = 3 or x = –4', 'x – 3 = 0 or x + 4 = 0'?

3. Formula$_2$ = +ARGUMENT$_1$ statement +L-LINK^2 truth function
| Formula | 410
+ARGUMENT$_2$ statement
| Formula

Here the argument slots may be filled with either Formula$_1$ or Formula$_2$, allowing us to build up complex formulas of arbitrary length:

$$x = 3 \text{ or } x = 3 \text{ or } x = 4 \text{ or } x = 0 \text{ or } \ldots \text{ or } x = 2.$$

'410' denotes a filler class of logical links, in which we want to include not only 'or' but also 'and', 'implies', 'iff', 'only if', 'if then', etc. For the 'if then', of course, we need an obligatory transformational rule shifting the ‘if’ to the head of the formula. Alternatively, we could write a separate syntagmeme:

4. Formula$_3$ = +L-LINK^2 –PART part of a truth function
| < if >

+ARGUMENT$_1$ statement +L-LINK^2 –PART part of a truth function
| Formula | < then >

+ARGUMENT$_2$ statement
| Formula

A separate syntagmeme is also needed for ‘not’:

5. Formula$_4$ = +L-LINK^1 truth function + ARGUMENT$_1$ statement
| < not; it is not the case that > | Formula

To complete the analysis of the clause level and below in (1), we must analyze the types of terms. How do we describe grammatically ‘$x^2 + x - 12$’, ‘0’, ‘x^2’, ‘$x - 3$’, ‘$(x - 3)$’, etc.?

6. Term$_1$ = $\pm_1$ MARGIN punctuator +ARGUMENT$_1$ element
| <(> | Term

+F-LINK^2 function +ARGUMENT$_2$ element
| *Function*^{2} | *Term*

$\pm_1$ MARGIN punctuator
| <)>

(6) is intended to describe constructions like ‘$(x - 3)$’ and ‘$(x + 4)$’, where a two-argument function ‘–’ or ‘+’ operates on terms (‘x’, ‘3’, ‘4’). The ‘$\pm_1$’ is intended to indicate that margin slots may or may not occur with filler parentheses, but if either of the two margin slots is filled, the other one *must* be (e.g., ‘$x - 3$’ and ‘$(x - 3)$’ are terms, but ‘$(x - 3$’ and ‘$x - 3)$’ are not).

In our corpus (1) we also find one term ‘x^2’ constructed with a unary function ‘$-^2$’ = “squared”. We can write this:

7. Term$_2$ = +ARGUMENT$_1$ element ± F-LINK^1 power

Term Numeral Variable	Superscript

We now describe the "word classes" referred to by the lowest level class words 'Numeral', 'Variable', 'Function^2', and 'Superscript' in (7).

Variable:

$x, x', x'', x_1, x_2, \ldots$

$y, y', y'', y_1, y_2, \ldots$

$z, z', z'', z_1, z_2, \ldots$

. . .

Superscript:

$-^2$	'squared'
$-^3$	'cubed'
$-^4$	'to the fourth power'
	. . .

Function^2:

+	'plus'
–	'minus'
·	'times'
/	'divided by'

Numeral:

0	'zero'
1	'one'
2	'two'
–1	'minus one'
	. . .
1.2	
1.23	
3.5	
	. . .

Of course, we could optionally write a further analysis of 'Variable' and 'Numeral'.

Variable = +NUCLEUS possible variable element

| variable letter ('x', 'y', 'z')

± MARGIN differentiation

subscripts
primes

(For example, 'x'' is 'x', a variable letter, juxtaposed with ''', a prime).

Numeral = ± F-LINK^1 negative +ARGUMENT positive number

<–>	unsigned numeral

8. Unsigned numeral = + (± DET$_1$ whole number ± DET$_2$

digit sequence	< . >

decimal point ± DET$_3$ fractional part
| digit sequence

(The first '+' indicates that *one* of DET$_1$ or DET$_2$ = determiners-one or -two must occur.)

digit sequence = + DET$_1$ specifier ± DET$^{(m)}$ specifier
| digit | digit

Here the superscript '$-^{(m)}$' is a special symbol indicating that the slot 'DET' may occur any number of times. This corresponds to the fact that strings of digits of arbitrary finite length m+1 are supposed to be 'digit sequences'.

digit: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

subscripts:	primes:
$-_1$ 'sub-one'	$-'$ 'prime'
$-_2$ 'sub-two'	$-''$ 'double-prime'
. . .	. . .

We need additional syntagmemic rules to specify (i) that the "whole number" part of an unsigned numeral (8) does not begin with zero (yet in some discourse contexts entries like '0.12' may be acceptable or even preferable to '.12'; (ii) that the "marginal" parentheses of (6) sometimes occur obligatorily (e.g., '(x + 3) • 4' to distinguish it from 'x + 3 • 4') and sometimes are virtually always absent (e.g., 'x + 3 = 0', not '(x + 3) = 0') in accordance with grammatico-contextual conditioning factors.

Note that we have not included the possibility of using parentheses as punctuators in the logical structure connecting atomic formulas (does 'x = 0 or x = 3 and y = 4' mean '(x = 0 or x = 3) and y = 4' or 'x = 0 or (x = 3 and y = 4)'?). In more complicated situations than elementary algebra some form of logical punctuation, whether or not by parentheses, must certainly be included in the descriptions of syntagmemes.

With some minor alterations, the above tagmemic constructions can also be used to describe formulas in the formalized language of a first-order theory. For first order theories, we would, in addition to specifications like (2), (3), (5), (6), and (7) above, have to include quantification:

Formula$_5$ = +MARGIN punctuator ±QUANTIFIER existential vs. universal quantification
| <(> |<∃>

+DET variable element	+MARGIN punctuator
Variable	<)>

±$_1$MARGIN punctuator	+NUCLEUS statement	±$_1$MARGIN punctuator
<(>	Formula	<)>

(Example: $(\exists\ x)(x + 3 = 0)$.)

n-adic predicates:

Formula$_6$ =	+P-LINK^n predicate^n	+ARGUMENT$_1$ element
	Predicate symbol^n	Term

+ . . .	+ARGUMENT$_n$ element
	Term

(Example: $p^{(n)}\ x_1\ x_2 \ldots x_n$.)

and perhaps n-ary functions:

Term$_3$ =	+F-LINK^n function^n	+ARGUMENT$_1$ element
	Function symbol^n	Term

+ . . .	+ARGUMENT$_n$ element
	Term

(Example: $f^{(n)}\ x_1\ x_2 \ldots x_n$.)

The possible formulas of elementary algebra can also be described quite compactly in terms of transformational grammar.

Initial symbol: S.
Intermediate symbols: T (term); L^1, L^2 (logical links); F^1, F^2 (function links); V (variable); N (numeral).
Terminal symbols: =, or, and, implies, . . . , x, y, z, 0, 1, 2, −1, . . . , +, −, ·, /, (,), . . .

Rules:

$S \rightarrow T = T$ (cf. (2))
$S \rightarrow S\ L^2\ S$ (cf. (3))
$S \rightarrow L^1\ S$ (cf. (5))
$L^2 \rightarrow$ or, and, implies, iff, only if, if then
$L^1 \rightarrow$ not (cf. (5))
$T \rightarrow T\ F^2\ T$ (cf. (6))
$T \rightarrow (T\ F^2\ T)$ (cf. (6))
$F^2 \rightarrow +, -, \cdot, /$
$T \rightarrow T\ F^1$ (cf. (7))
$F^1 \rightarrow \_^2, \_^3, \_^4, \ldots$
$T \rightarrow V$
$T \rightarrow N$
$V \rightarrow x, y, z, x', y', z', x_1, y_1, z_1, \ldots$
$N \rightarrow 0, 1, 2, -1, 1.2, 1.23, 3.5, \ldots$

3. FROM THE CLAUSE LEVEL TO THE PARAGRAPH LEVEL

The language of elementary algebra, like the language of formal first-order theories, has a definite structure above the clause (or "formula") level, as we shall see. Since transformational grammar has not yet significantly extended its techniques beyond the sentence, we will once more use tagmemic theory for our analysis, making an effort to separate clearly, as in § 2, between the grammatical structure of the language (in this case, the articulatory order of clauses) and the sememic structure (in this case, the relations among the referents in a mathematical model or models).

Simple Mathematical Sentence (SMS) = +NUCLEUS statement
| Formula

+DET assertional
| "closure" (suprasegmental)

In this formulation a simple sentence is a formula not embedded in any larger formula: it is a formula "plus" the non-localized contextual marker "closure". Closure may be, but need not be, marked explicitly by a period or (in more formalized contexts) '$\vdash$'. For example, in (1) '$x - 3 = 0$' and '$x + 4 = 0$' are formulas, but '$x - 3 = 0$ or $x + 4 = 0$' is a SMS as well as a formula.

9. Complex Mathematical Sentence (CMS) = ±MAR
comments
reasons
explanations

Dependent Clause
Dependent Phrase
Independent English Sentence
. . .

+NUCLEUS assertion ±MAR
SMS

comments
reasons
explanations

Dependent Clause
Dependent Phrase
Independent English Sentence
. . .

In (9) we envision the possibility that comments may be added to a derivation like (1), as follows:

10. Problem: solve $x^2 + x - 12 = 0$
Solution: let $x^2 + x - 12 = 0$
then $(x - 3) \cdot (x + 4) = 0$
hence $x - 3 = 0$ or $x + 4 = 0$
$x = 3$ or $x = -4$.

or more profusely:

11. Find all real numbers x such that $x^2 + x - 12 = 0$.
Solution: $x^2 + x - 12$ is assumed.
By algebra, this is true iff $(x - 3) \cdot (x + 4) = 0$.
Now the product of two factors is 0 iff one of the factors is zero, by law . . .
Hence $x - 3 = 0$ or $x + 4 = 0$.
This is true iff $x = 3$ or $x = -4$.
Hence, finally, $x^2 + x - 12 = 0$ iff $x = 3$ or $x = -4$.

All the English words except for logical links are to be included in the margin slots of (9).

Next, a *proof* is a construction consisting of at least one CMS followed optionally by any number m ($\geqslant$ 0) of further CMS's.

$$\begin{array}{ll} \text{12. Proof} = +\text{DET semantic implication} & \pm\text{DET}^{(m)}\text{ semantic implication} \\ \qquad\qquad\quad \big|\ \text{CMS} & \big|\ \text{CMS} \end{array}$$

But the syntagmeme (12) is far from capturing the real genius of a proof. On the naive level, we can say in addition

1. that the first CMS in a proof usually reiterates the hypothesis or hypotheses of the theorem proved;
2. that the last CMS in a proof states either the theorem as a whole, the conclusion which one wanted to follow from the hypotheses, or something closely related to the theorem or problem posed;
3. that though the intermediate CMS's in a proof may deviate considerably in content from the statement of the theorem, yet the different CMS's follow one another in an orderly and rigidly prescribed way.

4. FOUR INTERLOCKING HIERARCHIES

To deal in more detail with the characteristics of proofs, we find it convenient to exploit the tagmemic concept (Pike, 1967; Chapter 15) of interlocking hierarchies: phonological, grammatical, and sememic (the last not clearly separated from the lexical in Pike). In the case of written material, we speak of (1) a graphical, instead of phonological, hierarchy, defined in relation to the physical and perceptual material on the page, as the product of human formative activity; (2) a grammatical hierarchy defined in relation to the articulatory structure and ordering of graphical elements into well-formed, meaningful wholes; (3) a sememic hierarchy defined in relation to the conceptual content referred to and communicated by the grammatical structure; (4) a pragmatic hierarchy (by other tagmemic writers combined with the sememic hierarchy or left in residue) defined in relation to the purpose of the author to convince his readers, entertain them, exhort them, etc. These hierarchies appear, of course, at any level of discourse (morpheme, word, phrase, sentence, paragraph, dialogue), but in our case illustrations (to be exhibited in § 5) are most illuminating at the proof level.

The nature of a proof depends heavily upon the type of discourse in which one is engaged. Graphical, grammatical, sememic, and pragmatic characteristics of the discourse each influence the types of construction, or syntagmemes, used in forming a proof, and the choice among potential "fillers" of the tagmemic slots. Thus:

A. *Graphically*, in the choice of fillers we are influenced by the commercially available type sets as well as by the conventions established earlier in the discourse; in the choice of graphical construction types, we are influenced by the available display techniques and the linear nature of written discourse.

B. *Grammatically*, in the choice of fillers we have analytic or synthetic, inductive or deductive proof arrangements; in the choice of construction types we are influenced by discourse-determined rules of derivation.

C. *Sememically*, or algebraically, in the choice of fillers we have several kinds of algebraic manipulation that achieve the same result; in the choice of construction types we are influenced by what algebraic laws are taken as axiomatic in the discourse.

D. *Pragmatically*, in the choice of fillers, we have different types of proof according to the mathematical background and intelligence of the readers and the purpose of the discourse; and similar influence is exerted on the construction types, i.e., on the arrangements of the material.

Furthermore, the graphical, grammatical, sememic, and pragmatic factors exert their influence not independently, but interlockingly (cf. Pike, 1967; Chapter 15).

5. ILLUSTRATED PROOF TYPES

Now let us illustrate, using variants of our problem (1), how each of the four factors – graphical, grammatical, sememic, and pragmatic – influence the fillers and constructions in a proof. Under each heading – graphical fillers, graphical constructions, grammatical fillers, grammatical constructions, etc. – we exhibit some alternate forms of (1) which, so far as possible, leave everything in (1) fixed except the particular aspect (e.g., the graphical filler) in question. Because of the interlocking of hierarchies, of course, some variation in the other aspects is unavoidable, but at any one time our focus is on one particular aspect and its variation.

A1. *Graphical fillers*

a. In elementary algebra, 'x', 'y', and 'z' usually represent variables and 'a', 'b', and 'c' constants, though this choice is discourse-dependent. Illustration 1 might or might not be permissible, depending on discourse context.

b. Once a variable letter is chosen, it must denote the same sememe for the duration of a proof.

Illustration 1

Solve: $a^2 + a - 12 = 0$

$(a - 3) \cdot (a + 4) = 0$

$a - 3 = 0$ or $a + 4 = 0$

$a = 3$ or $a = -4$

We may not write a proof as in Illustration 2. Note at this point the interlocking of the graphical and sememic hierarchies.

c. Greek letters, italic letters, capital letters, etc., may or may not be available, and may or may not be invested with special significance. Illustration 3, with capital letters 'A', 'B', and 'C', might be used to say tacitly, "This is a canonical form".

A2. *Graphical constructions*

a. Proofs may be a sequence of lines (cf. (1)), a sequence of two lines (Illustration 4), or a tree (Illustration 5), depending on discourse. Note here that the graphical possibilities interlock closely with discourse-established grammatical conventions on the structure of proofs.

b. The number of distinct types of parenthesis, (, [, { , etc., and the graphical means for indicating subordination of clauses, place practical limits on the degree of (grammatical) embedding. Is Illustration 6 or 7 readable?

B1. *Grammatical fillers*

The order and logical techniques of proof may vary. A proof may be strictly in reverse order (Illustration 8), of the "iff" type (Illustration 9), or hypothetical (Illustration 10).

B2. *Grammatical constructions*

The validity of a proof depends on what rules of derivation are allowed by the discourse, and how much previous information the reader possesses and is allowed to draw on. In an advanced physics book, Illustration 11 is an acceptable proof. If algebraic manipulation (the sememics) is in focus, one might need a proof like Illustration 12 for completeness; if logic (the grammar) is in focus, one might find Illustration 13 or (again depending on discourse) even a derivation using only the rule "modus ponens" and a limited set of axiom schemata.

C1. *Sememic fillers*

Different algebraic methods – factorization (cf. (1)), completing the square (Illustration 14), the quadratic formula (Illustration 15), Newton's method, the "guess" method (Illustration 16), etc. – can be used to solve the same problem.

C2. *Sememic constructions*

The kind and amount of detail that must be supplied in proving '$x^2 + x - 12 = 0$ iff $x = 3$ or $x = -4$' is dependent on the axioms and theorems assumed in the discourse. If all algebraic identities are axioms, as is more or less the case in a textbook on theoretical physics, Illustration 11 is an adequate proof.[5] If, on the other hand, one's starting point is the axioms of an algebraically closed field, one might want a proof such as Illustration 17. One might even want to show in still greater detail than in Illustration 17 that $4 \cdot (-3) = -12$ or $1 = (-3) + 4$. If one's starting point is Peano's axioms or set-theoretic axioms, the proof would be even more laborious.

D1. *Pragmatic fillers*

The kind of (sememic) method used depends on the purpose of the discourse. Is one trying to teach facility at algebraic manipulation? Then teach completing the square (as in Illustration 14) and factoring (as in (1)). Is one trying to give a simple method applicable to every case? Then teach the quadratic formula (as in Illustration 15). Is one trying to illustrate the meaning of the variable 'x' in '$x^2 + x - 12 = 0$'? Teach the "guess" method (as in Illustration 16). Is one trying to give a method for the engineer to use on n-degree equations? Teach Newton's method.

D2. *Pragmatic constructions*

a. The *amount* of explanation depends on how much can be filled in by the reader.[6] Thus, for less able high school freshmen, one might provide Illustration 18.

b. The *type* of information given depends on the readers and the purpose. One may *explain*, as in Illustration 18, or one may *question*, as in Illustration 19.

Illustration 2

$$x^2 + x - 12 = 0$$
$$(y - 3) \cdot (y + 4) = 0$$
$$y - 3 = 0 \text{ or } y + 4 = 0$$
$$z = 3 \text{ or } z = -4$$

Illustration 3

$$Ax^2 + Bx + C = 0$$
$$A \cdot (x - (-B + \sqrt{(B^2 - 4AC)}) /2A))(x + (-B - \sqrt{(B^2 - 4AC)}) /2A) = 0$$
. . .

Illustration 4

1. $x^2 + x - 12 = 0$	Given.
2. $(x - 3) \cdot (x + 4) = 0$	Factor 1.
3. $x - 3 = 0$ or $x + 4 = 0$	Apply properties of zero divisors to 2.

4. $x = 3$ or $x + 4 = 0$ Add '3' to '$x - 3 = 0$' in 3.
5. $x = 3$ or $x = -4$ Add '-4' to '$x + 4 = 0$' in 4.

Illustration 5

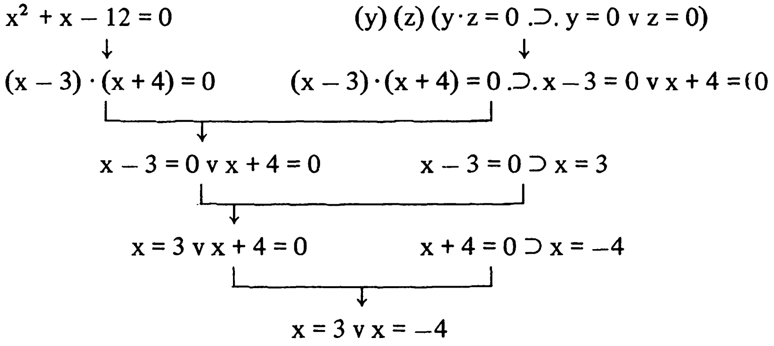

Illustration 6

$$((((x) \cdot (x)) + ((1) \cdot (x))) + (-12)) = ((((1) \cdot (x)) + (-3)) \cdot ((x) + (+4))))$$

Illustration 7

If $x^2 + x - 12 = 0$ then if if $x^2 + x - 12 = 0$ then $(x - 3) \cdot (x + 4) = 00$ then if $x - 3 \neq 0$ then $x = -4$.

Illustration 8

$x = 3$ or $x = -4$ follows from
$x - 3 = 0$ or $x + 4 = 0$ follows from
$(x - 3) \cdot (x + 4) = 0$ follows from
$x^2 + x - 12 = 0$

Illustration 9

$x^2 + x - 12 = 0$ iff $(x - 3) \cdot (x + 4) = 0$ iff $x - 3 = 0$ or $x + 4 = 0$ iff $x = 3$ or $x = -4$.

Illustration 10

Suppose $x \neq 3$.
Then $x^2 + x - 12 = 0$ implies
$(x - 3) \cdot (x + 4) = 0$ implies
$(x - 3)^{-1} \cdot (x - 3) \cdot (x + 4) = 0$ implies
$x + 4 = 0$ implies
$x = -4$.
Hence $x^2 + x - 12 = 0$ implies $x = -4$.
Hence, whatever x is, even if $x = 3$,
$x^2 + x - 12 = 0$ implies $x = 3$ or $x = -4$.

Illustration 11

$x^2 + x - 12 = 0$
$x = 3$ or $x = -4$.

Illustration 12

$x^2 + x - 12 = 0$
$x = 1 \cdot x$
$1 = (-3) + 4$
$x = ((-3) + 4) \cdot x$
$x^2 + ((-3) + 4) \cdot x - 12 = 0$
$x^2 + ((-3) \cdot x + 4 \cdot x) - 12 = 0$
$(x^2 + (-3) \cdot x) + (4 \cdot x - 12) = 0$
$(x \cdot x + (-3) \cdot x) + (4 \cdot x + (-12)) = 0$
$(x \cdot x + x \cdot (-3)) + (4 \cdot x + 4 \cdot (-3)) = 0$
$x \cdot (x + (-3)) + 4 \cdot (x + (-3)) = 0$
$(x + 4) \cdot (x + (-3)) = 0$
. . .

Illustration 13

1.	$x^2 + x - 12 = 0$	hypothesis
2.	$(x - 3) \cdot (x + 4) = 0$	from 1. by algebra
3.	$(y)(z)(y \cdot z = 0 \;.\!\supset\!.\; y = 0 \vee z = 0)$	postulate or theorem
4.	$(x - 3) \cdot (x + 4) = 0 \;.\!\supset\!.\; x - 3 = 0 \vee x + 4 = 0$	instantiation of 3
5.	$x - 3 = 0 \vee x + 4 = 0$	modus ponens from 2,4
6.	$\quad x - 3 = 0$	hypothesis
7.	$\quad x = 3$	algebra from 6
8.	$\quad x = 3 \vee x = -4$	disjunction introduction from 7

9.	$x + 4 = 0$	hypothesis
10.	$x = -4$	algebra from 9
11.	$x = 3 \vee x = -4$	disjunction introduction from 10
12.	$x = 3 \vee x = -4$	from 5, 6, 8, 9, 11 by "disjunction elimination"

Illustration 14

$$x^2 + x - 12 = 0$$
$$x^2 + x = 12$$
$$x^2 + x + 1/4 = 49/4$$
$$(x + 1/2)^2 = (7/2)^2$$
$$x + 1/2 = \pm 7/2$$
$$x + 1/2 = 7/2 \text{ or } x + 1/2 = -7/2$$
$$x = 3 \text{ or } x = -4.$$

Illustration 15

$$x^2 + x - 12 = 0$$
$$x = (-(+1) \pm \sqrt{((+1)^2 - 4(+1)(-12))})/2(+1) = (-1 \pm \sqrt{(49)})/2 = (-1 \pm 7)/2 = 3 \text{ or } -4.$$

Illustration 16

$x^2 + x - 12 = 0.$	What is x?	
Try $x = 1.$	$1^2 + 1 - 12 \neq 0.$	
Try $x = 2.$	$2^2 + 2 - 12 \neq 0.$	
Try $x = 3.$	$3^2 + 3 - 12 = 0.$	3 is a solution.
. . .		
Try $x = -3.$	$(-3)^2 + (-3) - 12 \neq 0.$	
Try $x = -4.$	$(-4)^2 + (-4) - 12 = 0.$	-4 is a solution.
. . .		

Illustration 17

$x^2 + x - 12 = 0$ is an abbreviation for

$(x^2 + x) + (-12) = 0$	
$(x^2 + 1 \cdot x) + (-12) = 0$	(because $1 \cdot x = x$)
$(x^2 + ((-3) + 4) \cdot x) + (-12) = 0$	(because $1 = (-3) + 4$)
$(x^2 + ((-3) \cdot x + 4 \cdot x)) + (-12) = 0$	(distributivity)

$((x^2 + (-3) \cdot x) + 4 \cdot x) + (-12) = 0$ (associativity)
$(x^2 + (-3) \cdot x) + (4 \cdot x + (-12)) = 0$ (associativity)
$(x^2 + x \cdot (-3)) + (4 \cdot x + (-12)) = 0$ (commutativity)
$(x^2 + x \cdot (-3)) + (4 \cdot x + 4 \cdot (-3)) = 0$
$x \cdot (x + (-3)) + 4 \cdot (x + (-3)) = 0$ (distributivity)
. . .

Illustration 18

Solve $x^2 + x - 12 = 0$, where x is an unknown number.
$(x - 3) \cdot (x + 4) = 0$
(When you multiply, $(x - 3) \cdot (x + 4) = x \cdot (x + 4) + (-3) \cdot (x + 4) = x^2 + 4x + (-3)x + (-12) = x^2 + x - 12$.)
$x - 3 = 0$ or $x + 4 = 0$
(If a product of two numbers $x - 3$ and $x + 4$ is 0, one of the two must be 0.)
. . .

Illustration 19

$x^2 + x - 12 = 0$
(Which of the methods – factoring, completing the square, quadratic formula – should we try first?)
Try factoring.
$(x - 3) \cdot (x + 4) = 0.$
(How did we know to try $x - 3$ and $x + 4$? What can we say about each factor $x - 3$ and $x + 4$ separately?)
$x - 3 = 0$ or $x + 4 = 0.$
(Why?)
$x = 3$ or $x = -4.$
(Why? How would you describe the operation of going from $x - 3 = 0$ to $x = 3$?)

6. ABOVE THE PARAGRAPH LEVEL

Tagmemic formulas like (2) can be written to approximate some of the discourse structure *above* the level of the proof (the mathematical paragraph). At such a level the structure is highly conditioned by pragmatic factors: is a book to be a how-to book, a textbook, a self-teaching book, a supplement to lectures, an exercise book? Is it to be of high or low mathematical sophistication? Because of its more formal style, a textbook designed, say, for upperclass or graduate mathematics majors is actually easier to describe than most of the other types of mathematics books. Nevertheless, since on the higher levels we must begin to deal with the less-formalized English that accompanies the proofs, the difficulties multiply and our descriptions must inevitably be more approximate.

The next larger unit above a proof is a theorem.

Theorem = +INTRODUCTION statement to be proved
    | CMS

+BODY proof
    | Proof

A theorem with proof *preceding* its statement (a not uncommon stylistic variant in some texts) might be considered as a pragmatically conditioned allosyntagma of the same syntagmeme.

Theorem Body = ±MARGIN illustration +NUCLEUS
    | Example
Comment | Theorem

deduction ±MARGIN explanation
    | Example
    | Comment
    | Informal Deducation
    | Second Theorem Body (an alternate proof of the same result)

Section = +BODY result ±BODY$^{(m)}$ result ±MARGIN application
    | Theorem Body | Theorem Body | Set of exercises

Chapter = ±INTRODUCTION preliminaries
    | Introd. section

+NUCLEUS result ±NUCLEUS$^{(m)}$ result ±MARGIN application
    | Section | Section | Set of exercises

7. SEMIOTICAL REMARKS

I regard it as no accident that it is possible to analyze proofs in terms of graphical, grammatical, sememic, and pragmatic hierarchies on the one hand and in terms of syntagmemes and fillers in the other. I expect that a complete analysis of any given unit of discourse involves, at least implicitly, an analysis in terms of each of the boxes of Figure 2. Figure 2 can be further elaborated if one recognizes that each hierarchy possesses not only syntagmemes (construction types) and fillers but also items and tagmemes (slot-filler correlation), as in Figure 3.

For the purpose of general philosophical analysis, the pragmatic and graphical aspects can be further broken down, as is done in detail in P. Verburg (1965). Cf. Figure 4.

Though this last division may not be immediately useful from a narrowly linguistic point of view, it *is* useful in bringing to the fore the connection between linguistic studies and sociological, psychological, biological, and physical studies of oral and written mathematical communication.

FIGURE 2

	syntagmemes	filler classes
graphical/phonological		
grammatico-syntactic		
sememic		
pragmatic		

FIGURE 4

aspect		syntagmemes	filler classes	items	tagmemes
pragmatic	. . .				
	aethetic				
	economic				
	social				
	sememic				
	grammatical				
phono-logical	psychical (or perceptual)				
	biotic				
	physical				
	kinematic				
	. . .				

FIGURE 3

aspect	syntag-memes	filler-classes	items	tagmemes
pragmatic	amount of detail; arrangement types	classes of methods and goals	a proof as representing a social goal plus method	slots for pragmatic methods
sememic	choice of axioms leading to different proof-forms	classes of algebraic methods	a single method; a particular proof as a sememic whole	slots for algebraic methods
grammatical	choice of rules of derivation leading to different syntactic forms of proof	variant orders and techniques of proof (class of analytic proofs, class of synthetic proofs, etc.)	a particular proof as a grammatical whole	grammatical slots for proofs
graphical	graphical forms of proof	classes of graphical strings (those with or without capital letters, italics, etc.)	a particular string of letters as a graphical whole	graphical slots for graphical strings

NOTES

1. For an introduction to the field of formal languages, cf. M. Gross and A. Lentin (1970).
2. Most past work, like A. Bentley's (1932) and the work of logicians, has been more philosophical and logical than linguistic. Computational linguistics is closer to the mark. But our interest is in dealing with mathematics *in situ*, as it comes from the pens of mathematicians, not as it is adapted and modified for computer use.
3. The "exotic" links and g-links are thus purely etic from the standpoint of analysis of elementary algebra. Only when one considers the problem of analyzing arbitrary new formal languages, as in Carnap (1951; Part IV), does the etic classification come into prominence.
4. A somewhat more precise formulation would delete this rule and add a transform-

ation rule which optionally or obligatorily deletes parentheses in certain well-defined contexts.

5. When Illustration 11 is used as an example of a *grammatically* influenced variant of (1), focus is on the possibility of its arising from a loosened concept of proof (proof being a syntactic = grammatical concept). On the other hand, when the same illustration is used as an example of *sememic* influence, focus is on the set of axioms as they define a set of models (either the integers alone, all algebraically closed fields, or even all fields). A similar difference applies to Illustration 12 vs. Illustration 17. Grammatical and sememic influences interlock without becoming identical.

6. The pragmatic influence on "amount of information" interlocks with, but is not identical with, the sememic influence. Pragmatically, one asks, "What will the reader understand?"; sememically, "How much has been (axiomatically) assumed?"

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